Extensional Vibration of Axisymmetrical Shells

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The paper deals with the axisymmetrical extensional vibration of thin shells of revolution and with arbitrary contour and constant thickness. A set of equations are established which describe the axisymmetrical motion of the shell considering only the membrane stresses. For the special case of spherical shells, the equations are reduced to the dynamic equations of thin spherical shells given by Love. For a shell of arbitrary contour, a numerical solution is used where the extensional vibration data of a thin ellipsoidal shell are given as an illustrative example. In the second part of the paper, frequency and vibration mode equations are established for hemispherical shells attached to an elastic structure through a bulkhead ring.

Nomenclature

r_0	= radius of the shell element measured from the symmetrical axis
r_1, r_2	= radii of curvature along the meridian and the circumferential directions
R	= radius of the spherical shell
a, b	= semimajor and semiminor axes of the ellipsoidal shell
e	= $(a^2 - b^2)^{1/2}/a$ = eccentricity of the basic ellipse
h	= shell thickness
ϕ	= meridian angle
N_ϕ, N_θ	= normal forces in the shell
e_ϕ, e_θ	= normal strains in the shell
E, ν	= Young's modulus and Poisson's ratio
ρ	= density of the shell material
u, v, w	= displacements along the meridian, circumferential and normal directions
u^*, v^*, w^*	= amplitudes of displacements u, v, w corresponding to a certain frequency
t	= time
ω	= circular frequency
Ω	= $(\rho/E)^{1/2}\omega a$ = a nondimensional frequency
K	= spring constant of the bulkhead support
M	= mass of the bulkhead ring per unit length along the shell circumference
k	= $KR(1 - \nu^2)/Eh$ = nondimensional spring constant
m	= $M(1 - \nu^2)/\rho hR$ = nondimensional bulkhead ring mass
A_n	= nominal amplitude of deformation
$P_n(Z)$	= Legender function of the first kind and of degree n

Introduction

DURING the launch phase of a multistage missile, the shell structure in the upper stages is under substantial acceleration. Portions of the liquid fuel and oxidizer tank shells are stretched by the accelerative force. Subsequently, the booster rocket cutoff and the thrust decay cause a deceleration, whereby the strain energy stored in the tank shell structure is released and is transformed into kinetic energy. The resulting dynamic response of the tank structure may involve undesirable transient motion that affects the performance of the upper stage rockets.

The shells involved in the tank structure are very thin and have a radius/thickness ratio in the range of 500–1000. The stresses and strains in the shell are essentially of the membrane type with localized bending stresses near the edge where the shell is attached to a bulkhead ring or other strengthening members. The paper represents an initial study of the problem and deals with the axisymmetrical extension vibration of a shell of revolution and with arbitrary contour.

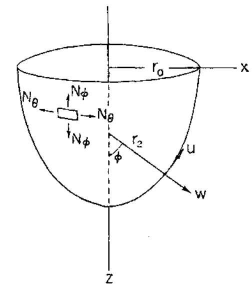

Fig. 1 An axisymmetrical shell.

In the general case of the axisymmetrical shell, a numerical solution is used to study the shell dynamic behavior. For a hemispherical shell with bulkhead attachment, frequency equations and the corresponding mode shapes are established explicitly.

Equations of Motion

For an axisymmetrical shell of arbitrary contour (Fig. 1), the following force equilibrium equations are established along the meridian and the normal directions:

$$\frac{\partial}{\partial\phi}(N_\phi r_0) - N_\theta r_1 \cos\phi = r_1 r_0 \rho h \frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$(1/r_1)N_\phi + (1/r_2)N_\theta = -\rho h(\partial^2 w/\partial t^2) \tag{2}$$

The stress-strain relations and the expressions for the strain components are introduced:

$$N_\phi = [Eh/(1 - \nu^2)](e_\phi + \nu e_\theta) \tag{3}$$

$$N_\theta = [Eh/(1 - \nu^2)](e_\theta + \nu e_\phi) \tag{4}$$

$$e_\phi = (1/r_1)(\partial u/\partial\phi) + w/r_1 \tag{5}$$

$$e_\theta = (u/r_2)\cot\phi + w/r_2 \tag{6}$$

For a given frequency ω, the u, w displacements are represented by

$$u = u^*(\phi)e^{i\omega t} \tag{7}$$

$$w = w^*(\phi)e^{i\omega t} \tag{8}$$

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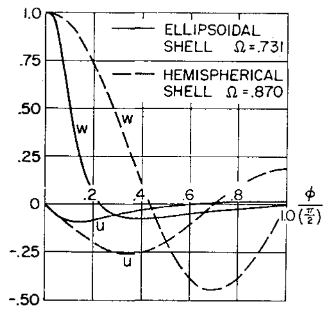

Fig. 2 Normal modes of the ellipsoidal and the hemispherical shells, first mode, first sequence.

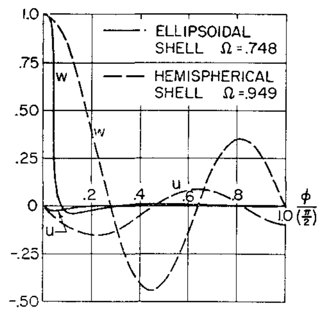

Fig. 3 Normal modes of the ellipsoidal and the hemispherical shells, second mode, first sequence.

Substituting Eqs. (3-8) into Eqs. (1) and (2), the following dynamic equations for the axisymmetrical shell are reached:

$$\begin{aligned}&\frac{1}{r_1^2}\frac{d^2u^*}{d\phi^2} + \left(\frac{1}{r_1r_2}\cot\phi - \frac{1}{r_1^3}\frac{dr_1}{d\phi}\right)\frac{du^*}{d\phi} + \\ &\left[-\frac{1-\nu}{r_2^2}\cot^2\phi - \frac{\nu}{r_1r_2^2}\cot\phi\frac{dr_2}{d\phi} - \frac{\nu\csc^2\phi}{r_1r_2} + \frac{(1-\nu^2)\Omega^2}{a^2}\right]u^* + \left(\frac{\nu}{r_1r_2}+\frac{1}{r_2^2}\right)\frac{dw^*}{d\phi} + \\ &\left[(1-\nu)\left(\frac{1}{r_1r_2}-\frac{1}{r_2^2}\right)\cot\phi - \frac{\nu}{r_1r_2^2}\frac{dr_2}{d\phi} - \frac{1}{r_1^3}\frac{dr_1}{d\phi}\right]w^* = 0\end{aligned} \quad (9)$$

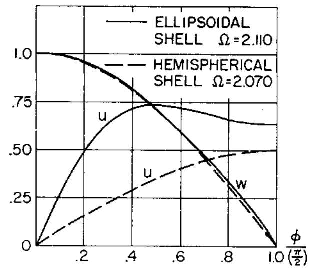

Fig. 4 Normal modes of ellipsoidal and the hemispherical shells, first mode, second sequence.

$$\begin{aligned}w^* = -\frac{1}{(1/r_1^2) + (2\nu/r_1r_2) + (1/r_2^2) - (1-\nu^2)(\Omega^2/a^2)} \times \\ \left[\left(\frac{1}{r_1^2}+\frac{\nu}{r_1r_2}\right)\frac{du^*}{d\phi} + \left(\frac{\nu}{r_1r_2}+\frac{1}{r_2^2}\right)\cot\phi u^*\right]\end{aligned} \quad (10)$$

Differentiating both sides of Eq. (10) with respect to ϕ yields

$$\begin{aligned}\frac{dw^*}{d\phi} = -\frac{1}{(1/r_1^2) + (2\nu/r_1r_2) + (1/r_2^2) - (1-\nu^2)(\Omega^2/a^2)} \times \\ \left[\left(\frac{1}{r_1^2}+\frac{\nu}{r_1r_2}\right)\frac{d^2u^*}{d\phi^2} - \left(\frac{2}{r_1^3}\frac{dr_1}{d\phi} + \frac{\nu}{r_1^2r_2}\frac{dr_1}{d\phi} + \frac{\nu}{r_1r_2^2}\frac{dr_2}{d\phi}\right)\frac{du^*}{d\phi} + \right. \\ \left(\frac{\nu}{r_1r_2}+\frac{1}{r_2^2}\right)\cot\phi\frac{du^*}{d\phi} - \left(\frac{\nu}{r_1^2r_2}\frac{dr_1}{d\phi} - \right. \\ \left.\left.\frac{\nu}{r_1r_2^2}\frac{dr_2}{d\phi} - \frac{2}{r_2^3}\frac{dr_2}{d\phi}\right)\cot\phi u^* - \left(\frac{\nu}{r_1r_2}+\frac{1}{r_2^2}\right)\csc^2\phi u^*\right] - \\ \frac{2}{[(1/r_1^2) + (2\nu/r_1r_2) + (1/r_2^2) - (1-\nu^2)(\Omega^2/a^2)]^2} \times \\ \left[\frac{1}{r_1^2}\left(\frac{1}{r_1}+\frac{\nu}{r_2}\right)\frac{dr_1}{d\phi} + \frac{1}{r_2^2}\left(\frac{\nu}{r_1}+\frac{1}{r_2}\right)\frac{dr_2}{d\phi}\right] \times \\ \left[\left(\frac{1}{r_1^2}+\frac{\nu}{r_1r_2}\right)\frac{du^*}{d\phi} + \left(\frac{\nu}{r_1r_2}+\frac{1}{r_2^2}\right)\cot\phi u^*\right]\end{aligned} \quad (11)$$

Equations (10) and (11) are substituted into Eq. (9) to eliminate w^*. After a certain amount of simplification, the following equation is reached:

$$\begin{aligned}&(1-\nu^2)\frac{1}{r_1^2}\left(\frac{1}{r_2^2}-\frac{\Omega^2}{a^2}\right)\frac{d^2u^*}{d\phi^2} + \left\{\left[\frac{1}{r_1r_2}\left(\frac{1}{r_1^2}+\frac{2\nu}{r_1r_2}+\frac{1}{r_2^2}-(1-\nu^2)\frac{\Omega^2}{a^2}\right) - \left(\frac{\nu}{r_1r_2}+\frac{1}{r_1^2}\right)\left(\frac{1}{r_1r_2}+\frac{\nu}{r_2^2}\right)\right]\cot\phi + \right. \\ &\left(\frac{\nu}{r_1r_2}+\frac{1}{r_1^2}\right)\left(\frac{3}{r_1^3}\frac{dr_1}{d\phi} + \frac{\nu}{r_1^2r_2}\frac{dr_1}{d\phi} + \frac{2\nu}{r_1r_2^2}\frac{dr_2}{d\phi}\right) - \frac{1}{r_1^3}\frac{dr_1}{d\phi}\left[\frac{1}{r_1^2}+\frac{2\nu}{r_1r_2}+\frac{1}{r_2^2}-(1-\nu^2)\frac{\Omega^2}{a^2}\right] - \\ &\left.\frac{2\left(\frac{\nu}{r_1r_2}+\frac{1}{r_2^2}\right)^2\left[\left(\frac{1}{r_1}+\frac{\nu}{r_2}\right)\frac{1}{r_1^2}\frac{dr_1}{d\phi} + \left(\frac{\nu}{r_1}+\frac{1}{r_2}\right)\frac{1}{r_2^2}\frac{dr_2}{d\phi}\right]}{\frac{1}{r_1^2}+\frac{2\nu}{r_1r_2}+\frac{1}{r_2^2}-(1-\nu^2)\frac{\Omega^2}{a^2}}\right\}\frac{du^*}{d\phi} + \left\{\left[-\left(\frac{1-\nu}{r_2^2}+\frac{\nu}{r_1r_2}\right)\times\right.\right. \\ &\left.\left[\frac{1}{r_1^2}+\frac{2\nu}{r_1r_2}+\frac{1}{r_2^2}-(1-\nu^2)\frac{\Omega^2}{a^2}\right] + \left(\frac{\nu}{r_1r_2}+\frac{1}{r_2^2}\right)\left(\frac{1}{r_1^2}-\frac{1-2\nu}{r_1r_2}+\frac{1-\nu}{r_2^2}\right)\right]\cot^2\phi + \\ &\left[-\frac{\nu}{r_1r_2}\left[\frac{1}{r_1^2}+\frac{2\nu}{r_1r_2}+\frac{1}{r_2^2}-(1-\nu^2)\frac{\Omega^2}{a^2}\right]\frac{dr_2}{d\phi} + \left(\frac{\nu}{r_1r_2}+\frac{1}{r_1^2}\right)\left(\frac{\nu}{r_1^2r_2}\frac{dr_1}{d\phi}+\frac{\nu}{r_1r_2^2}\frac{dr_2}{d\phi}+\frac{2}{r_2^3}\frac{dr_2}{d\phi}\right) + \right. \\ &\left.\left(\frac{\nu}{r_1r_2}+\frac{1}{r_2^2}\right)\left(\frac{\nu}{r_1r_2^2}\frac{dr_2}{d\phi}+\frac{1}{r_1^3}\frac{dr_1}{d\phi}\right) - \frac{2\left(\frac{\nu}{r_1r_2}+\frac{1}{r_1^2}\right)\left(\frac{\nu}{r_1r_2}+\frac{1}{r_2^2}\right)\left[\left(\frac{1}{r_1}+\frac{\nu}{r_2}\right)\frac{1}{r_1^2}\frac{dr_1}{d\phi}+\left(\frac{\nu}{r_1}+\frac{1}{r_2}\right)\frac{1}{r_2^2}\frac{dr_2}{d\phi}\right]}{\frac{1}{r_1^2}+\frac{2\nu}{r_1r_2}+\frac{1}{r_2^2}-(1-\nu^2)\frac{\Omega^2}{a^2}}\right]\cot\phi - \\ &\left.\left[\frac{\nu}{r_1r_2}-(1-\nu^2)\frac{\Omega^2}{a^2}\right]\left[\frac{1}{r_1^2}+\frac{2\nu}{r_1r_2}+\frac{1}{r_2^2}-(1-\nu^2)\frac{\Omega^2}{a^2}\right] + \left(\frac{\nu}{r_1r_2}+\frac{1}{r_2^2}\right)\left(\frac{\nu}{r_1r_2}+\frac{1}{r_2^2}\right)\right\}u^* = 0\end{aligned} \quad (12)$$

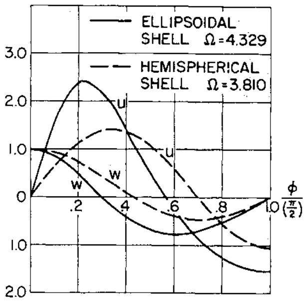

Fig. 5 Normal modes of the ellipsoidal and the hemispherical shells, second mode, second sequence.

Equations (10) and (12) govern the extensional vibration of an axisymmetrical thin shell. For a nontruncated shell, certain terms in the equations become indeterminate at $\phi = 0$, in which case, the following relations may be used:

$$r_1 = r_2 \qquad dr_1/d\phi = dr_2/d\phi = 0$$

$$u^* = dw^*/d\phi = 0 \qquad \cot\phi\, u^* = du^*/d\phi$$

Equations (10) and (12) are then reduced to

$$w^* = \frac{2}{2 - (1-\nu)\Omega^2\,(r_1{}^2/a^2)}\,\frac{du^*}{d\phi} \tag{13}$$

$$d^2u^*/d\phi^2 = 0 \tag{14}$$

For the special case of a spherical shell, the following relations are true at any location ϕ:

$$r_1 = r_2 = a = R \qquad dr_1/d\phi = dr_2/d\phi = 0$$

in which case, Eqs. (10) and (12) are reduced to

$$\frac{d^2u^*}{d\phi^2} + \cot\phi\,\frac{du^*}{d\phi} + \left\{ -\cot^2\phi + \frac{(1+\nu) - [\nu - (1-\nu^2)\Omega^2][2 - (1-\nu)\Omega^2]}{(1-\nu)(1-\Omega^2)} \right\} u^* = 0 \tag{15}$$

$$w^* = \frac{1}{(1-\nu)\Omega^2 - 2}\left(\frac{du^*}{d\phi} + \cot\phi u^*\right) \tag{16}$$

An Ellipsoidal Shell

For an ellipsoidal shell where the semiminor axis b coincides with the axis of symmetry, the two radii of the shell at location ϕ are

$$r_1 = \frac{a(1-e^2)}{(1 - e^2\cos^2\phi)^{3/2}} \tag{17}$$

$$r_2 = \frac{a}{(1 - e^2\cos^2\phi)^{1/2}} \tag{18}$$

Based on Eqs. (17) and (18), the gradients $dr_1/d\phi$, $dr_2/d\phi$ can be found easily. With the radii data, numerical integration technique can be applied to integrate Eqs. (10) and (12) for a given frequency Ω. The integration starts with an assumed nominal value of w^* at $\phi = 0$. Eqs. (13) and (14) are used. Beyond $\phi = 0$, Eqs. (10) and (12) are used to cover the whole shell. At the shell terminal, certain boundary conditions are to be satisfied. An unsatisfied boundary condition indicates the assumed frequency Ω is not a natural frequency of the shell structure. The integration is repeated with a different frequency. Using a high-speed digital computer, the natural frequencies and the corresponding normal modes can be determined without much difficulty. In the illustrative example, an ellipsoidal shell has an axes ratio of $b/a = 0.75$ so that the eccentricity e is $\frac{1}{4}\,(7)^{1/2}$. The shell terminates at $\phi = \frac{1}{2}\pi$ as a free edge where we require that $N_\theta = 0$, which implies that $w^* = 0$, $du^*/d\phi = 0$.

Numerical integration yields the frequencies listed in Table 1. The corresponding normal modes are plotted in Figs. 2–5. Also given are the corresponding data for a hemispherical shell with a free edge.

For a hemispherical shell with a free edge, it has been found that the extensional vibration modes can be classified into two sequences.[1, 2] In the low-frequency sequence, $0 < \Omega < 1.0$, an infinite number of modes are located. The high-frequency sequence covers the frequency range $1.0 < \Omega < \infty$. The lowest frequency value of the high sequence for a hemispherical shell is $\Omega = 2.070$. In a recent study on spherical shells,[3] in which the bending of the shell element is also considered, it is found that the membrane theory yields fairly accurate frequency and mode shape data except in the neighborhood of $\Omega = 1.0$. In other words, in the low sequence of the hemispherical shell based on the membrane theory, only the first few modes have true physical meaning. The remaining modes of the low sequence have frequencies which are crowded in a region near $\Omega = 1.0$ and are different from the modes based on the combined extensional and bending theory. The modes in the high-frequency sequence based on the membrane theory are compatible with data obtained using the combined extensional and bending theory.

Referring to Figs. 2–5, it is noted that there is a one-to-one correspondence between the mode shapes for the ellipsoidal shell and the hemispherical shell. In the low-frequency sequence, the w deformation of the ellipsoidal shell is concentrated in the region near the axis of symmetry. It should be noted that the result is valid only for very thin shells where the shell resistance to bending is negligible. Otherwise, the bending of the shell element, corresponding to a mode shape such as is shown in Fig. 3, is substantial and has to be considered. It should also be noted that the deformation data are plotted against ϕ. For an ellipsoidal shell, the data are compressed in the neighborhood of $\phi = 0$, as compared with a plot where the deformation data are plotted against the meridian length. In the case of the forced response of an axisymmetrical shell, it may be necessary to consider only the first few modes in the low frequency sequence. In which case, the relatively small amplitude of the u displacement makes it justifiable to ignore the u-displacement in a first approximation. Axisymmetrical shells with other types of edge conditions can be handled in a similar manner through the integration of Eqs. (10) and (12).

A Hemispherical Shell

The equations of extensional vibration of a spherical shell are given in Eqs. (15) and (16). It can be shown that they are equivalent to the equations given by Love for the thin spherical shell.[1,2] Love's equations are listed below:

$$\frac{d^2w^*}{d\phi^2} + \cot\phi\,\frac{dw^*}{d\phi} + \frac{[2 - (1-\nu)\Omega^2][1 + (1+\nu)\Omega^2]}{1 - \Omega^2}\,w^* = 0 \tag{19}$$

$$u^* = \left[(1-\Omega^2)/[1 + (1+\nu)\Omega^2]\right](dw^*/d\phi) \tag{20}$$

The solution of either pair of equations, (15) and (16) or (19) and (20), for a nontruncated spherical shell is

$$w^* = A_n P_n(\cos\phi) \tag{21}$$

$$u^* = A_n\,\frac{1-\Omega^2}{1 + (1+\nu)\Omega^2}\,\frac{d}{d\phi}\,P_n\,(\cos\phi) \tag{22}$$

where n is given by the following equation:

$$n(n+1) - 2 = [(1+\nu)\Omega^2/(1-\Omega^2)][3 - (1-\nu)\Omega^2] \tag{23}$$

Table 1 Natural frequency Ω data

	1st sequence	2nd sequence
Ellipsoidal shell $[e = \frac{1}{4}\,(7)^{1/2}]$	0.731 0.748 ...	2.110 4.327 ...
Hemispherical shell	0.870 0.949 ...	2.070 3.810 ...

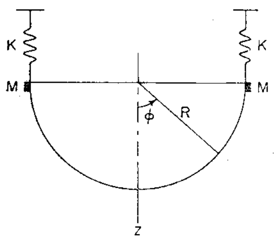

Fig. 6 Sectional view of a spring-supported hemispherical bulkhead.

Consider a hemispherical shell attached to a spring-supported bulkhead ring as shown in Fig. 6. First the bulkhead ring is assumed to be free to expand. Force balance along the axis of symmetry at the edge of the hemispherical shell requires that

$$N_\phi = -Ku - M\ddot{u} \qquad \phi = \tfrac{1}{2}\pi$$

Making use of Eqs. (3, 5, 6, 21, 22), the preceding condition can be rewritten as

$$\frac{d^2}{d\phi^2} P_n(0) + (k - m\Omega^2) \frac{d}{d\phi} P_n(0) + \frac{(1+\nu)\,[1 + (1+\nu)\,\Omega^2]}{1 - \Omega^2} P_n(0) = 0 \quad (24)$$

which is equivalent to

$$P_n(0) - \frac{k - m\Omega^2}{(1-\nu)[1 + (1+\nu)\,\Omega^2]} \frac{d}{d\phi} P_n(0) = 0 \quad (25)$$

Equations (23) and (25) determine the natural frequencies of the spring-supported hemispherical shell. Again, the solution is reached by feeding in a large number of frequency values Ω. The frequency that satisfies both Eqs. (23) and (25) corresponds to a natural frequency of the supported system. The shell normal modes are given by Eqs. (21) and (22). Table 2 gives the natural frequencies and the corresponding n values for a combination of various values of k, m.

In Table 2, the data in the first column correspond to a hemispherical shell with a free edge so that $k = m = 0$. Typical vibration mode shapes for a spring-supported hemispherical shell are given in Figs. 7 and 8.

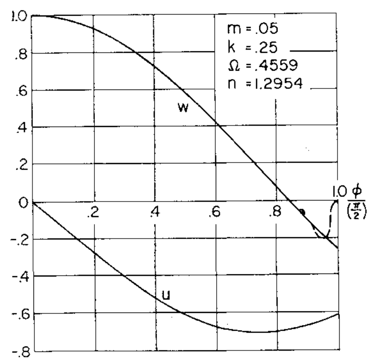

Fig. 7 The first normal mode of a spring-supported hemispherical shell. (The dotted line indicates the edge effect correction).

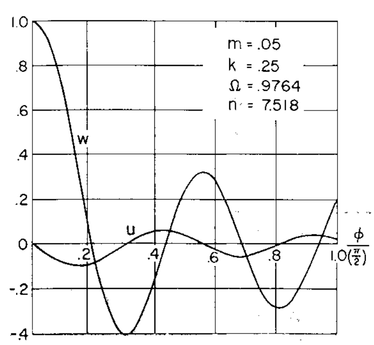

Fig. 8 The fourth normal mode of a spring-supported hemispherical shell.

Table 2 Natural frequency data for a spring-supported hemispherical shell

	$k = 0$, $m = 0$	$k = 0.25$, $m = 0$	$k = 0.25$, $m = 0.05$	$k = 0.50$, $m = 0$	$k = 1.00$, $m = 0$
First sequence					
Ω	...	0.4631	0.4558	0.5762	0.6560
n	...	1.3060	1.2954	1.5098	1.7142
Ω	0.870	0.8954	0.8923	0.9071	0.9177
n	3.000	3.3825	3.323	3.611	3.931
Ω	0.949	0.9574	0.9563	0.9603	0.9623
n	5.000	5.494	5.427	5.710	5.874
Ω	0.973	0.9768	0.9764	0.9778	0.9784
n	7.000	7.582	7.519	7.737	7.861
Second sequence					
Ω	2.070	2.1041	2.0749	2.1392	2.2084
n	1.000	1.0545	1.0078	1.0940	1.2115
Ω	3.810	3.8499	3.7377	3.8886	3.9628
n	3.000	3.0399	2.9266	3.077	3.152
Ω	5.840	5.871	5.689	...	...
n	5.000	5.028	4.845	...	...

The preceding analysis assumes that the bulkhead ring may breath freely, since w^* does not vanish at $\phi = 1/2\pi$. The ratio between the two displacement components at the shell edge is

$$w^*/u^* = (k - m\Omega^2)/[(1-\nu)(1-\Omega^2)] \quad (26)$$

In actual case, the bulkhead ring is so rigid that

$$w^* \to 0 \qquad dw^*/d\phi \to 0 \text{ at } \phi = 1/2\pi \quad (27)$$

It is not possible to satisfy the additional boundary conditions (27) when the solution based on the membrane theory is used. Evidently, the bending of the shell element has to be considered in order to satisfy the additional conditions. In a low frequency mode, it may be reasonable to limit the bending to the part of the shell near the bulkhead ring. This approach is somewhat similar to the edge effect solution used in the static shell analysis. The detail of this approach will be presented in a separate paper.

Conclusion

The present paper established the equations governing the axisymmetrical extensional vibrations of shells of revolution. Frequency equations are presented for shells attached to an elastic support. In order to study the exact effect of local bending on the vibration modes, dynamic equations including shell bending terms are to be used. For very thin shells, the data presented in the paper serve as a starting point toward further study of the problem.

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